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A new impedance-based approach to analysis and control of sound scattering

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Abstract

A new theoretical solution to the problem of scattering sound by arbitrary obstacles in a fluid medium is presented. The scattering matrices are expressed in terms of three input impedance matrices defined with respect to the fluid–scatterer interface: one impedance matrix describes the scatterer and two matrices characterize the fluid. General properties of the scattering matrices are studied. Some useful representations of the scattered field are proposed. The general solution is applied to three well-known practical problems for which some new results are obtained or the existing results are generalized. For the first problem (of the most efficient passive sound absorber), an equation that determines its surface impedances is derived. A similar result is obtained for the second problem (of the best scatterer). For the third problem (of how to make a body to be acoustically transparent), "active" solutions that can be realized as smart coatings are proposed. Computer simulations illustrate the results.

1. Introduction

The problem of scattering sound by obstacles, e.g. by elastic bodies in fluids or cavities in solids, has been a subject of interest for many years. This is due to its numerous practical applications such as acoustic tomography, fault detection, acoustic radars, etc. Analytical solutions to the scattering boundary value problem are obtained only for obstacles of simple geometries (plane, cylinder, sphere). For other cases, a lot of approximate methods have been developed each having a certain range of applicability depending on obstacle and medium properties, incident fields, the way of observing the scattered field, frequency range, and on other problem parameters—see, e.g. Refs. [1–8]. The main interest of most studies is backscattering of incident plane waves by elastic obstacles in a fluid medium.

This paper is oriented at a comparatively new problem of the scattering theory—the problem of active control of scattered fields by means of thin-layered smart coatings. Such coatings, containing distributed or/and discrete sensors and actuators, are placed on the interface between the scatterer and medium and operate using the data measured at this surface. The theory presented is based on the introduction of characteristics that are defined just on the scatterer–medium interface. These characteristics are three input impedance

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matrices: one matrix describes vibrations of the scatterer and two matrices characterize the fluid vibration. It is shown that the full solution of the scattering problem can be written in terms of these three matrices.

The paper consists of two parts—theory and applications. The main theoretical results are given in Sections 2–4. In Section 2, the scattering matrices are introduced and expressed through the three impedance matrices. Solutions for particular scatterers, such as rigid and soft bodies, are given in Section 3. Some new general representations of the scattered field are presented in Section 4.

Usefulness of the proposed impedance-based approach is demonstrated in the second part of the paper (Sections 5–7) where it is applied to three practical scattering problems. The first of them is the problem of the most efficient passive sound absorber. It is considered in Section 5. The main result of this section is the following: a finite-size body is the best sound absorber if its matrix of the input surface in vacuo impedances is equal to the Hermitian conjugate of the matrix of the radiation impedances. The result does not depend on the incident field and is valid for bodies of any geometry in any acoustic environment. A numerical example of sound absorption by one and two air bubbles in water is presented. The second problem is considered in Section 6 where a definition of the ideal passive scatterer is given. It is shown that a body scatters maximum of the sound energy if its impedance matrix is purely imaginary (the body is lossless) and close to the matrix of the radiation reactances with opposite sign. Section 7 is devoted to the third problem—the problem of the acoustically transparent body, i.e. the problem of how to make a body in a fluid to be non-scattering and, hence, invisible to acoustic radars. It is shown that a body is acoustically transparent if its surface impedance matrix does not differ from that of the fluid in the volume of the body. This condition can be implemented by active means, e.g. using a smart coating on the body surface. An illustrative example with a cylindrical shell is considered.

2. Theory

2.1. Statement and assumptions

Consider a finite-size elastic body (scatterer, absorber, obstacle) of arbitrary geometry immersed in an acoustic fluid medium. The body occupies a volume V and has an outer surface A which is the interface with the fluid. The medium is not necessarily homogeneous and unbounded. It is deemed inviscid thus exerting only normal loads on the body surface. Both the body and fluid are assumed to obey linear differential equations. Time dependence of all field variables is assumed to be harmonic and exponential coefficient $\exp(-i\omega t)$ is abbreviated throughout the paper, ω being circular frequency.

In the fluid medium outside the body, some acoustic sources are present. In the absence of the body, they produce a pressure field with complex amplitude $p_i(x)$, which is called the incident field. In the presence of the body, the pressure field component $p_s(x)$ scattered by the body also appears. The total pressure field at a point with coordinates x in the exterior of the body is, thus, represented by the sum of these two components,

$$p(x) = p_i(x) + p_s(x). \tag{1}$$

The problem is stated as to determine the scattered field component $p_s(x)$ if given is the incident field component $p_i(x)$.

Unlike the commonly used approach (which consists in finding solutions that satisfy the governing equations and boundary conditions at all bounding surfaces including interface A), the solution below is obtained in terms of the impedance characteristics and scattering coefficients that are defined with respect to the fluid–body interface.

2.2. Impedance characteristics

To introduce the needed impedance characteristics, it is convenient to represent the surface A of the body as a set of N small-size surface elements ΔA_n, $\quad n = 1, 2, \ldots, N$. The number N is not fixed but the dimensions of the elements are assumed to be less than half of the fluid wavelength. The pressure, normal velocity and other field characteristics may then be taken as uniform within each element and the quantities continuously distributed over surface A may be represented on A as N-vectors. For example, the total pressure field, Eq. (1), and the corresponding fluid particle normal velocity $v(y)$ are represented on A as the following N-vectors

$$\mathbf{p} = [p(y_1)\Delta A_1, \ldots, p(y_N)\Delta A_N]^{\mathrm{T}}, \quad \mathbf{v} = [v(y_1), \ldots, v(y_N)]^{\mathrm{T}}. \tag{2}$$

The components of vector $\mathbf{p}$ are the complex amplitudes of the forces acting on areas ΔA_n and vector $\mathbf{v}$ consist of N complex amplitudes of the outward normal velocity, y_n being the coordinates of a point of the surface element ΔA_n. Similarly, N-vectors $\mathbf{p}_i$, $\mathbf{v}_i$ and $\mathbf{p}_s$, $\mathbf{v}_s$ are introduced to describe the incident and scattered fields on A.

Now let us define three basic square matrices of order N: $\mathbf{Z}$, $\mathbf{Z}_i$, and $\mathbf{Z}_r$ that are necessary for solving the scattering problem.

Matrix $\mathbf{Z}$ is the matrix of the input surface impedances of the elastic body in vacuo. If $\mathbf{f} = [f_1, \ldots, f_N]^{\mathrm{T}}$ is a vector of external forces applied to the surface elements ΔA_n and $\mathbf{u} = [u_1, \ldots, u_N]^{\mathrm{T}}$ is the normal velocity vector of the body surface response in vacuo, then, by definition, *the body impedance matrix* $\mathbf{Z}$ and its inverse, the mobility matrix, $\mathbf{Y} = \mathbf{Z}^{-1}$, relate these two vectors as

$$\mathbf{f} = \mathbf{Z}\mathbf{u} \quad \text{or} \quad \mathbf{u} = \mathbf{Y}\mathbf{f}. \tag{3}$$

The positive direction of the forces and velocities coincide with the outer normal vector to surface A. It will be clear from further consideration that matrix $\mathbf{Z}$ (or $\mathbf{Y}$) fully characterizes an elastic body as a scatterer.

Two impedance matrices, $\mathbf{Z}_i$ and $\mathbf{Z}_r$, are needed to characterize the fluid in the scattering problem. Both matrices are defined with respect to the closed surface A.

Matrix $\mathbf{Z}_i$ of *the internal impedances* of the fluid is defined similarly to the body impedance matrix $\mathbf{Z}$ and describes forced vibrations of the isolated volume V filled with fluid. "Isolated" means that the volume V is set apart from the external fluid (and placed in vacuum). Let $\mathbf{g}_1$ be a N-vector of external forces applied to surface A and $\mathbf{v}_1$ be the surface normal velocity response vector, defined as in Eq. (2). Then matrix $\mathbf{Z}_i$ and its inverse, the internal mobility matrix, $\mathbf{Y}_i = \mathbf{Z}_i^{-1}$, link the two vectors in the same manner as in Eq. (3)

$$\mathbf{g}_1 = \mathbf{Z}_i\mathbf{v}_1 \quad \text{or} \quad \mathbf{v}_1 = \mathbf{Y}_i\mathbf{g}_1. \tag{4}$$

Matrix $\mathbf{Z}_r$ of the *radiation impedances* is defined as a matrix of the input impedances of the fluid in the exterior of surface A isolated from the scatterer and with all the acoustic sources in it switched off. Here "isolated" means that the scatterer is removed and volume V contains vacuum. If $\mathbf{g}_2$ is an N-vector of forces applied to A from inside the empty volume V and $\mathbf{v}_2$ is vector of N complex amplitudes of the normal velocity responses of A to these forces, matrix $\mathbf{Z}_r$ and its inverse, the radiation mobility matrix, $\mathbf{Y}_r = \mathbf{Z}_r^{-1}$, link them as

$$\mathbf{g}_2 = \mathbf{Z}_r\mathbf{v}_2 \quad \text{or} \quad \mathbf{v}_2 = \mathbf{Y}_r\mathbf{g}_2. \tag{5}$$

All vectors in Eqs. (3)–(5) are positive when directed along the outer normal vector to A.

The three impedance and mobility matrices defined in Eqs. (3)–(5) are assumed to be complex-valued and symmetric with respect to the main diagonal. Physically, the symmetry means that the reciprocity theorem is valid in the elastic body and fluid. In particular, this means that they are linear, do not contain gyroscopic elements, and forces of the Lorentz and Coriolis types are excluded from their governing equations. For scatterers of simple geometries in homogeneous unbounded fluid the matrices may be obtained analytically. In general case, they can be computed using one of the available numerical methods, e.g. FEM for $\mathbf{Z}$ and $\mathbf{Z}_i$ [9], and BEM for $\mathbf{Z}_r$ [10].

Using these matrices, the following equations can be written for the incident and scattered components of the total field (1) on interface A:

$$\mathbf{p}_i + \mathbf{Z}_i\mathbf{v}_i = 0, \tag{6}$$

$$\mathbf{p}_i + \mathbf{p}_s + \mathbf{Z}(\mathbf{v}_i + \mathbf{v}_s) = 0, \tag{7}$$

$$\mathbf{p}_s - \mathbf{Z}_r\mathbf{v}_s = 0. \tag{8}$$

To prove Eq. (6), consider the incident field in the fluid without the scatterer. In this case, the volume V filled with fluid is acted with force $[-p_i(y)]$ distributed on A, the normal velocity on A being equal to $v_i(y)$, $y \in A$. Discretizing the surface A and taking into account that in the first Eq. (4) $\mathbf{g}_1 = -\mathbf{p}_i$ and $\mathbf{v}_1 = \mathbf{v}_i$, one obtains Eq. (6). Similarly, in the body–fluid system, the force that acts on the body surface is equal to

$[-p(y)] = -[p_i(y) + p_s(y)]$ and the velocity response is $v(y) = [v_i(y) + v_s(y)]$. Eq. (7) is obtained after discretization of these functions as in Eq. (2) and substitution of relations $\mathbf{f} = -\mathbf{p}$ and $\mathbf{u} = \mathbf{v}$ into the first Eq. (3). To derive Eq. (8), one should consider the scattered field component alone, i.e. without the incident component. Since all the singularities (sources) of the scattered field lie inside the surface A, this field can be regarded as the field radiated by the surface A vibrating with the prescribed normal velocity $v_s(y)$ into the passive exterior of A, the function $p_s(y)$ being the radiated field pressure amplitude on A. After discretization of A, one obtains Eq. (8) from the first Eq. (5) using relations $\mathbf{g}_2 = \mathbf{p}_s$ and $\mathbf{v}_2 = \mathbf{v}_s$.

2.3. Scattering matrices

Let us introduce two square scattering matrices of order N, $\mathbf{S}$ and $\mathbf{Q}$, that relate the pressure and normal velocity of the scattered field on the interface A correspondingly to the pressure and normal velocity of the incident field on A in the form of linear equations

$$\mathbf{v}_s = \mathbf{Q}\mathbf{v}_i, \quad \mathbf{p}_s = \mathbf{S}\mathbf{p}_i. \tag{9}$$

Here N-vectors $\mathbf{p}_s$,$\mathbf{v}_s$ and $\mathbf{p}_i$,$\mathbf{v}_i$ are defined as in Eq. (2). Using relations (6)–(8) one can obtain the following equations for the scattering matrices

$$\begin{aligned}\mathbf{Q} &= (\mathbf{Z}_r + \mathbf{Z})^{-1}(\mathbf{Z}_i - \mathbf{Z}),\\ \mathbf{S} &= (\mathbf{Y}_r + \mathbf{Y})^{-1}(\mathbf{Y}_i - \mathbf{Y}),\end{aligned} \tag{10}$$

where $\mathbf{I}$ is the identity matrix of order N. Consequently, the total field on A is equal to

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_i + \mathbf{v}_s = (\mathbf{I} + \mathbf{Q})\mathbf{v}_i = (\mathbf{Z}_r + \mathbf{Z})^{-1}(\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{v}_i,\\ \mathbf{p} &= \mathbf{p}_i + \mathbf{p}_s = (\mathbf{I} + \mathbf{S})\mathbf{p}_i = (\mathbf{Y}_r + \mathbf{Y})^{-1}(\mathbf{Y}_r + \mathbf{Y}_i)\mathbf{p}_i.\end{aligned} \tag{11}$$

Since the pressure and normal velocity on A obey the boundary integral equation [3,8], the scattering matrices, $\mathbf{Q}$ and $\mathbf{S}$, are not independent. It can be verified using Eqs. (6)–(11) that they satisfy the matrix equations

$$\begin{aligned}&\mathbf{Z}_r\mathbf{Q} + \mathbf{S}\mathbf{Z}_i = 0,\\ &\mathbf{Z}(\mathbf{I} + \mathbf{Q}) - (\mathbf{I} + \mathbf{S})\mathbf{Z}_i = 0.\end{aligned} \tag{12}$$

Eq. (10) (representing one of the main results of this paper) express the scattering matrices through the impedance and mobility characteristics of the medium and scatterer. They give the solution to the scattering problem that is very similar, by the form and essence, to the well-known solution for the reflection coefficients obtained by C. Fresnel in the simplest case of plane waves reflecting from a plane interface between two fluid media [11]. The theory proposed in this paper can, thus, be regarded as a straightforward extension of the Fresnel's theory to the general case of the scattering problem. However, in general case, three impedance (mobility) matrices are needed instead of two in the Fresnel's case. The reason is that, as shown in the next section, in this particular case of two connected fluid half-spaces, the internal and radiation impedances, $\mathbf{Z}_i$ and $\mathbf{Z}_r$, coincide.

In the literature on scattering problems, the solutions in the form of scattering matrices are widely used. They are defined, however, with respect to certain incoming and outgoing waves, the scattering coefficients relating their amplitudes. Such solutions are useful when the scatterer geometry conforms the geometry of the wave fronts, as, e.g. in case of a plane interface and plane waves (the Fresnel's theory), or a scattering sphere and spherical waves as well as a scattering cylinder and cylindrical waves. Unfortunately, the list of the well-studied wave functions is restricted by these three. So, the definition of the scattering matrices given in Eq. (9) that is valid for scatterers of arbitrary geometry seems appropriate. Its usefulness will be demonstrated in the applications of the theory (see Sections 5–7).

Physical sense of the scattering matrices, $\mathbf{S}$ and $\mathbf{Q}$, introduced in Eq. (9) is the following. Element S_{jm} (or Q_{jm}) is equal to the pressure (or velocity) amplitude of the scattered field at mth surface element ΔA_m if the incident field is the sound beam ensonifying only jth surface element ΔA_j with the unit pressure (or velocity) amplitude. In case when the fluid–scatterer interface is a coordinate surface of one of the separable coordinate systems, the relation between the scattering matrices (9) and the traditional scattering matrices (defined with respect to the wave amplitudes) can, if necessary, be obtained using the vector transformation algebra [12].

It should be emphasized that the scattering matrices, **S** and **Q**, are defined in Eq. (9) on the scatterer surface A and, hence, solution (10) describes the scattered field on this surface only. For many practical problems, it is quite sufficient as, e.g. in the applications of Sections 5–7. The scattered field at any other point x of the fluid may, if needed, be computed through these boundary values with the help of the Helmholtz–Huygens integral [8]. In the discretized form this integral is written as

$$p_s(x) = \mathbf{G}(x|y)\mathbf{v}_s + \mathbf{G}'(x|y)\mathbf{p}_s.$$

Here $\mathbf{G}(x|y) = [G(x|y_1), \ldots, G(x|y_N)]$ is the row N-vector of the Green's functions of the fluid without the scatterer each component $G(x|y_n)$ being equal to the pressure amplitude at point x produced by an elementary (monopole) source located at point $y_n \in \Delta A_n$. The prime means the derivative with respect to coordinate along the outer normal vector at point y_n. Similarly, the sound field penetrated into the elastic scatterer can, if necessary, be computed using the Green's function of the scatterer.

2.4. Symmetry properties

Symmetry of a square matrix with respect to the main diagonal, i.e. its equality to the transposed matrix, is equivalent to the reciprocity property of the system, which is described by the matrix, when the input and output of the system are interchanged. Such is the input impedance matrix of a linear vibratory system. Its symmetry means that the classical reciprocity theorem is valid in the system: the response does not change if the driving point and observation point are interchanged.

In the scattering problem, an incident field may also be regarded as excitation and the scattered field—as response. Symmetry of a scattering matrix, thus, characterizes a kind of reciprocity property of the scatterer–fluid system. Normally, the scattering matrices (9), **Q** and **S**, (as well as the commonly used wave scattering, reflection and transmission, matrices [11]), are not symmetric, though the impedance and mobility matrices are symmetric. The reason is that the scattering matrices are usually defined with respect to the linear amplitudes of the field or waves. As shown in ref.[13], if one uses certain energy-like characteristics of the fields and waves, instead of their amplitudes, the reciprocity may be restored. In the mathematical language, this means that certain combinations of the scattering and impedance matrices become symmetric. In our case, the following matrix combinations containing the scattering matrices (9) are found to be symmetric

$$\begin{aligned}&(\mathbf{Z}_i - \mathbf{Z})\mathbf{Q}, (\mathbf{Z}_r + \mathbf{Z})\mathbf{Q}, \mathbf{Q}(\mathbf{Z}_r + \mathbf{Z}_i),\\&(\mathbf{Y}_i - \mathbf{Y})\mathbf{S}, (\mathbf{Y}_r + \mathbf{Y})\mathbf{S}, (\mathbf{Y}_r + \mathbf{Y}_i)\mathbf{S},\\&\mathbf{Z}_r\mathbf{Q}\mathbf{S}^{\mathbf{T}}, \mathbf{Z}_i\mathbf{S}^{\mathbf{T}}\mathbf{Q}.\end{aligned} \tag{13}$$

Symmetry of some of them can be easily verified with the help of Eqs. (9)–(12). Others can be proved using the method of Ref. [13]. Note that symmetry of matrices (13) is a sequence of the classical reciprocity theorem, i.e. symmetry of matrices $\mathbf{Z}$, $\mathbf{Z}_i$ and $\mathbf{Z}_r$, which is assumed to be valid in the fluid and scatterer.

3. Particular cases

Some particular cases of the general solution (10) to the scattering problem that are used further in this paper are considered in the section.

3.1. Plane boundary between two fluids

Below it is shown that the classical Fresnel's formulae for reflection of plane waves from a plane fluid–fluid boundary follow directly from solution (10).

Consider two half-spaces filled with fluids 1 and 2 having a plane interface that coincides with plane (ξ,η) of the Cartesian coordinate system (ξ,η,ζ). In fluid 1 ($\zeta<0$), a plane pressure wave

$$p_i(\xi,\zeta) = p_i \exp[\mathrm{i}k\xi + i(k_1^2 - k^2)^{1/2}\varsigma] \tag{14}$$

impinges on the boundary. Two other plane waves are outgoing from the boundary–reflected wave $p_r(\xi,\zeta)$ in fluid 1 and transmitted wave $p_t(\xi,\zeta)$ in fluid 2 ($\zeta>0$):

$$\begin{aligned} p_r(\xi,\zeta) &= p_r \exp[\mathrm{i}k\xi - i(k_1^2 - k^2)^{1/2}\varsigma],\\ p_t(\xi,\zeta) &= p_t \exp[\mathrm{i}k\xi + i(k_2^2 - k^2)^{1/2}\varsigma]. \end{aligned} \tag{15}$$

Here, k_1 and k_2 are the wavenumbers of fluids 1 and 2, k is the wave trace on the interface which, according to the Snell's law [1], is identical in all three plane waves.

In the terminology of the present paper, the half-space $\zeta<0$, filled with fluid 1, is "the medium". The half-space $\zeta>0$, filled with fluid 2, is "the scatterer", the plane $\zeta=0$ is "the interface A" and reflected wave (15) represents "the scattered field". As the fields (14), (15) and all the field quantities depend on ξ via factor $\exp(\mathrm{i}k\xi)$, the pressure/velocity ratios are identical at all points of the interface A and the impedance matrices are reduced, in this particular case, to scalar impedance quantities. For "the scatterer", the input surface impedance of the half-space 2 with respect to the external force distributed on A with density $f\exp(\mathrm{i}k\xi)$ is equal to

$$Z = \rho_2 c_2 \frac{k_2}{\sqrt{k_2^2 - k^2}}, \tag{16}$$

where $\rho_2 c_2$ is the characteristic impedance of fluid 2. Similarly for "the medium", both the internal impedance Z_i and the radiation impedance Z_r represent the input surface impedance of the half-space 1 with respect to the external force $g\exp(\mathrm{i}k\xi)$ distributed on A, which is equal to

$$Z_i = Z_r = \rho_1 c_1 \frac{k_1}{\sqrt{k_1^2 - k^2}}. \tag{17}$$

In notation of Fresnel, impedance (16) is Z_2 and impedance (17) is Z_1. When these impedances are substituted in solution (10), one obtains exactly the Fresnel's formulae [1,11].

3.2. Rigid body

If a scattering body has a rigid boundary surface A, its mobility matrix is the zero matrix, $\mathbf{Y}=0$. The scattering matrices (10) are, in this case, equal to

$$\mathbf{Q}_{\text{rig}} = -\mathbf{I}, \quad \mathbf{S}_{\text{rig}} = \mathbf{Z}_r\mathbf{Y}_i. \tag{18}$$

Consequently, the normal velocity and pressure amplitudes (11) of the total field on A are

$$\begin{aligned} \mathbf{v}_{\text{rig}} &= \mathbf{v}_i + \mathbf{v}_s = 0,\\ \mathbf{p}_{\text{rig}} &= \mathbf{p}_i + \mathbf{p}_s = (\mathbf{I} + \mathbf{Z}_r\mathbf{Y}_i)\mathbf{p}_i = -(\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{v}_i. \end{aligned} \tag{19}$$

It is seen from Eq. (19), that the relation between the total pressure and the incident pressure on the surface of a rigid scatterer depends on the relation between the radiation and internal impedances of the fluid. When the surface is plane, these impedances are identical and the incident pressure field doubles its amplitude.

3.3. Soft body

If a scatterer has a pressure–released boundary surface A, its impedance matrix is the zero matrix, $\mathbf{Z}=0$. Solution (10) gives the following scattering matrices

$$\mathbf{Q}_{\text{soft}} = \mathbf{Y}_r\mathbf{Z}_i, \quad \mathbf{S}_{\text{soft}} = -\mathbf{I}. \tag{20}$$

The normal velocity and pressure amplitudes of the total field on A are, in this case, equal to

$$\mathbf{v}_{\text{soft}} = (\mathbf{I} + \mathbf{Y}_r\mathbf{Z}_i)\mathbf{v}_i = -(\mathbf{Y}_r + \mathbf{Y}_i)\mathbf{p}_i,$$
$$\mathbf{p}_{\text{soft}} = \mathbf{p}_i + \mathbf{p}_s = 0. \tag{21}$$

Here again, as is seen from Eq. (21), the relation between the total normal velocity and the velocity of the incident field on the surface of a soft scatterer depends only on the impedance properties of the fluid. The total velocity is double incident velocity if a pressure–released boundary is plane.

There is a close relation between the field scattered by a rigid body and the field scattered by a soft body of the same geometry. One can verify using Eqs. (18) and (20) that the scattering matrices of these two scatterers relate to each other as

$$\mathbf{Q}^{\text{T}}_{\text{soft}}\mathbf{S}_{\text{rig}} = \mathbf{I}, \quad \mathbf{S}^{\text{T}}_{\text{soft}}\mathbf{Q}_{\text{rig}} = \mathbf{I}. \tag{22}$$

These equations mean that, for a given incident field, the scatterers fields of the two scatterers are, in certain sense, supplementary being inverse to each other: the smaller the field scattered by a soft body, the larger is the field scattered by the rigid body. This result reminds the Babinet's theorem of diffraction theory [3]. It also follows from the first Eq. (22) that the scattering matrices of these two bodies, $\mathbf{S}_{\text{rig}}$ and $\mathbf{Q}^{\text{T}}_{\text{soft}}$, commute in multiplication, $\mathbf{Q}^{\text{T}}_{\text{soft}}\mathbf{S}_{\text{rig}} = \mathbf{S}_{\text{rig}}\mathbf{Q}^{\text{T}}_{\text{soft}}$, and, therefore, have identical eigenvectors and inverse eigenvalues.

4. Useful field representations

In scattering theory, it is common practice to represent the total field as the sum of two components—the incident and scattered components—see Eq. (1).The incident field component is a solution of the boundary value problem for the medium under consideration with the specified sound sources but without the scatterer. The scattered field component is a solution to the radiation boundary value problem for the same medium with the primary sound sources switched off and the scatterer being the source of the radiation field. This representation of the total field, thus, splits the initial boundary value problem into two simpler boundary value problems.

In this section, three new useful representations of the total field are given. Each of them is the sum of two components that are solutions of two auxiliary boundary value problems that are simpler than the initial one. These representations are particular cases of the more general result obtained in paper [14] for vibro-acoustic fields in linear complex vibratory systems.

4.1. Representation 1

In this representation, the first field component corresponds to the blocked surface A and, therefore, is equal to the field scattered by the corresponding rigid body (19)

$$p(x) = p_{\text{rig}}(x) + p_1(x). \tag{23}$$

The second field component $p_1(x)$ describes, according to the general theory [14], the forced coupled vibration of the scatterer in the medium with the initial sources switched off under the action of the "blocked force" $f_b(y)$. The blocked force is the force with which the external medium acts on the blocked surface A in the first auxiliary boundary value problem. In the discretized form it is equal, according to Eq. (19), to

$$\mathbf{f}_b = -\mathbf{p}_{\text{rig}} = (\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{v}_i.$$

The N-vectors of the velocity and pressure amplitudes of the second field component on A are

$$\mathbf{v}_1 = (\mathbf{Z}_r + \mathbf{Z})^{-1}\mathbf{f}_b = (\mathbf{Z}_r + \mathbf{Z})^{-1}(\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{v}_i = \mathbf{Q}_1\mathbf{v}_i,$$
$$\mathbf{p}_1 = \mathbf{Z}_r\mathbf{v}_1 = -\mathbf{Z}_r\mathbf{Q}_1\mathbf{Y}_i\mathbf{p}_i = \mathbf{S}_1\mathbf{p}_i. \tag{24}$$

The total field on A is, thus, represented as the sum of the components (23):

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_i + \mathbf{Q}_{\text{rig}}\mathbf{v}_i + \mathbf{Q}_1\mathbf{v}_i,\\ \mathbf{p} &= \mathbf{p}_i + \mathbf{S}_{\text{rig}}\mathbf{p}_i + \mathbf{S}_1\mathbf{p}_i.\end{aligned} \tag{25}$$

Consequently, the scattering matrices may be written as

$$\begin{aligned}\mathbf{Q} &= \mathbf{Q}_{\text{rig}} + \mathbf{Q}_1 = -\mathbf{I} + (\mathbf{Z}_r + \mathbf{Z})^{-1}(\mathbf{Z}_r + \mathbf{Z}_i),\\ \mathbf{S} &= \mathbf{S}_{\text{rig}} + \mathbf{S}_1 = \mathbf{Z}_r\mathbf{Y}_i - \mathbf{Z}_r(\mathbf{Z}_r + \mathbf{Z})^{-1}(\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{Y}_i.\end{aligned} \tag{26}$$

One can verify, making use of Eqs. (10), (18), and (24), that both Eq. (26) are mathematical identities. Note that Representation 1, written in form (25), is actually the sum of three field components—the incident component, the component scattered by the rigid body, and the forced vibration component. Modifications of Representation 1 can be met in the literature, e.g. in Ref. [5].

4.2. Representation 2

The first auxiliary boundary value problem in this representation corresponds to the problem of sound scattering by a body with pressure-released surface A. The first component of the total field coincides, therefore, with the field scattered by the soft body (21):

$$p(x) = p_{\text{soft}}(x) + p_2(x). \tag{27}$$

The second field component $p_2(x)$ describes, according to Ref. [14], the forced coupled vibration of the scatterer in the medium with the initial sources switched off under the action of the "kinematic excitation" $v_{\text{kin}}(y)$, applied to A. The kinematic excitation is equivalent to action of two equal and oppositely directed distributed forces that are applied on A separately to the body and medium. The values of these forces are such that the relative normal velocity response of the body and medium is equal to $v_{\text{kin}}(y)$. The velocity $v_{\text{kin}}(y)$ of the kinematic excitation is determined in the first auxiliary boundary value problem and is equal to the minus velocity of the surface of the soft body. In the discretized form it equals (see Eq. (21))

$$\mathbf{v}_{\text{kin}} = -\mathbf{v}_{\text{soft}} = (\mathbf{Y}_r + \mathbf{Y}_i)\mathbf{p}_i. \tag{28}$$

The second field component in the representation (27) on A is, hence, equal to

$$\begin{aligned}\mathbf{p}_2 &= (\mathbf{Y}_r + \mathbf{Y})^{-1}\mathbf{v}_{\text{kin}} = (\mathbf{Y}_r + \mathbf{Y})^{-1}(\mathbf{Y}_r + \mathbf{Y}_i)\mathbf{p}_i = \mathbf{S}_2\mathbf{p}_i,\\ \mathbf{v}_2 &= \mathbf{Y}_r\mathbf{p}_2 = -\mathbf{Y}_r\mathbf{S}_2\mathbf{Z}_i\mathbf{v}_i = \mathbf{Q}_2\mathbf{v}_i.\end{aligned} \tag{29}$$

The total field on A is represented as

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_i + \mathbf{Q}_{\text{soft}}\mathbf{v}_i + \mathbf{Q}_2\mathbf{v}_i,\\ \mathbf{p} &= \mathbf{p}_i + \mathbf{S}_{\text{soft}}\mathbf{p}_i + \mathbf{S}_2\mathbf{p}_i,\end{aligned} \tag{30}$$

so that the scattering matrices may be written as

$$\begin{aligned}\mathbf{Q} &= \mathbf{Q}_{\text{soft}} + \mathbf{Q}_2 = \mathbf{Y}_r\mathbf{Z}_i - \mathbf{Y}_r(\mathbf{Y}_r + \mathbf{Y})^{-1}(\mathbf{Y}_r + \mathbf{Y}_i)\mathbf{Z}_i,\\ \mathbf{S} &= \mathbf{S}_{\text{soft}} + \mathbf{S}_2 = -\mathbf{I} + (\mathbf{Y}_r + \mathbf{Y})^{-1}(\mathbf{Y}_r + \mathbf{Y}_i).\end{aligned} \tag{31}$$

Again, one can verify, using Eqs. (10), (20), and (29), that Eq. (31) are mathematical identities. Representation 2 in form (30) is in fact the sum of three field components.

4.3. Representation 3

This is the most general representation of the sound field scattered by a body with impedance matrix $\mathbf{Z}$. It consists of two components

$$p(x) = p_0(x) + \Delta p(x). \tag{32}$$

The first component $p_0(x)$ corresponds to the field scattered by a body of the same interface A geometry but with a reference surface impedance matrix $\mathbf{Z}_0$ that differs from $\mathbf{Z}$ by a complex symmetric matrix $\Delta\mathbf{Z}$:

$$\mathbf{Z} = \mathbf{Z}_0 + \Delta\mathbf{Z}. \tag{33}$$

The second component $\Delta p(x)$, corresponds to the forced coupled vibrations of the passive (i.e. without the initial sources) body–medium system under the action of either the force

$$\mathbf{f} = -\Delta\mathbf{Z}\mathbf{v}_0, \tag{34}$$

or the kinematic excitation

$$\mathbf{v}_{\text{kin}} = -\Delta\mathbf{Y}\mathbf{p}_0, \tag{35}$$

applied to surface A. Here $\Delta\mathbf{Y} = \mathbf{Y} - \mathbf{Y}_0$ is the difference between the mobility matrices of the scatterers (note that $\Delta\mathbf{Y}$ is not equal to $\Delta\mathbf{Z}^{-1}$). The vectors $\mathbf{v}_0$ and $\mathbf{p}_0$ in Eqs. (34) and (35) are the discretized normal velocity and pressure of the first component (32) on A. It can be verified that both the force excitation (34) and kinematic excitation (35) are equivalent in that they produce the same field component $\Delta p(x)$ in Eq. (32).

Representation (32) can be written in terms of the scattering matrices as follows

$$\mathbf{Q} = \mathbf{Q}_0 + \Delta\mathbf{Q}, \quad \mathbf{S} = \mathbf{S}_0 + \Delta\mathbf{S}, \tag{36}$$

where matrices $\mathbf{Q}_0$, $\mathbf{S}_0$ and $\mathbf{Q}$, $\mathbf{S}$ correspond to the scatterers with the impedance matrices $\mathbf{Z}_0$ and $\mathbf{Z}$, respectively. The difference matrices $\Delta\mathbf{Q}$ and $\Delta\mathbf{S}$ in Eq. (36) are equal to

$$\begin{aligned} \Delta\mathbf{Q} &= -(\mathbf{Z}_r + \mathbf{Z})^{-1}(\mathbf{Z} - \mathbf{Z}_0)(\mathbf{Z}_r + \mathbf{Z}_0)^{-1}(\mathbf{Z}_r + \mathbf{Z}_i), \\ \Delta\mathbf{S} &= -(\mathbf{Y}_r + \mathbf{Y})^{-1}(\mathbf{Y} - \mathbf{Y}_0)(\mathbf{Y}_r + \mathbf{Y}_0)^{-1}(\mathbf{Y}_r + \mathbf{Y}_i). \end{aligned} \tag{37}$$

All known representations of the scattered field are particular cases of the representation (36), (37). When the reference mobility matrix $\mathbf{Y}_0$ is the zero matrix (the reference body is rigid), Eqs. (36) and (37) reduce to Eqs. (26) of Representation 1. When the reference impedance matrix $\mathbf{Z}_0$ is the zero matrix (soft body), they yield Eqs. (31) of Representation 2. When the reference body impedance matrix is equal to the fluid internal matrix, $\mathbf{Z}_0 = \mathbf{Z}_i$ (the situation of the medium without a scatterer), the scattering matrices, $\mathbf{Q}_0$ and $\mathbf{S}_0$, are zero matrices, and Eqs. (36) and (37) become identical to Eqs. (10). This particular case is, thus, equivalent to the classical representation (1).

5. The best passive absorber

As the first application of the general solution (9), (10) a boundary condition of the impedance type is obtained for the best passive absorber, i.e. for a body that, among all bodies of the same configuration, absorbs maximum of the incident field energy.

In optics, ideal absorption is associated with the "black body" notion introduced by G. Kirchhoff in 1859. This is "a body that absorbs all the light energy which falls upon it" [15]. By definition, the absorption cross-section, as well as the scattering cross-section, of a black body is always equal to its geometrical cross-sectional area. Black bodies do not exist in Nature but there are several models (e.g. a body with local surface impedance ρc or the model of Kirchhoff [16] for which the total field is equal to the incident field on the illuminated part of the body surface and it is zero on the shadow part) that describe the main black body property (non-reflection) in the very high frequency limit where the body dimensions in terms of wavelengths tend to infinity.

In acoustics and electrodynamics, the absorber dimensions are usually compared with wavelengths. In this frequency range, a notion of the best absorber is also useful. The problem of the best passive absorber was first formulated and solved in Ref. [17] in terms of the amplitudes of incoming and outgoing (cylindrical) waves. It was shown there that the best absorber fully absorbs all the incoming waves so that the total field does not contain outgoing waves.

Below, the problem of the best absorber is reformulated and a new solution is obtained in terms of the surface impedance properties. It is shown in this section that the input impedance matrix of the best absorber is equal to Hermitian conjugate of the radiation impedance matrix. It is also demonstrated in illustrative examples that, even in the simplest cases, the absorption cross-section of the best absorber is several orders of magnitude greater than that of the classical black body of Kirchhoff.

5.1. Impedance matrix of the best absorber

Consider an absorber of sound, i.e. a finite-size body in a medium, subject to an incident field. The sound power entering the absorber is, by definition, equal to

$$\Phi = -\tfrac{1}{2}\,\mathrm{Re}(\mathbf{v}^*\mathbf{p}) = \tfrac{1}{2}\,\mathrm{Re}(\mathbf{v}^*\mathbf{Z}\mathbf{v}) = \tfrac{1}{2}\mathbf{v}^*\mathbf{R}\mathbf{v} = \tfrac{1}{2}\mathbf{v}_i^*(\mathbf{I}+\mathbf{Q}^*)\mathbf{R}(\mathbf{I}+\mathbf{Q})\mathbf{v}_i, \tag{38}$$

where $\mathbf{v}$ and $\mathbf{p}$ are the normal velocity and pressure vectors as defined in Eq. (2), $\mathbf{Z} = \mathbf{R} + \mathbf{i}\mathbf{X}$ is an in vacuo impedance symmetric matrix of the absorber, $\mathbf{R}$ and $\mathbf{X}$ being, correspondingly, the resistance and reactance matrices. The absorbed power (38) is a scalar real-valued non-negative quantity though the total and incident velocity amplitudes, $\mathbf{v}$ and $\mathbf{v}_i$, may be complex-valued. By using the first Eq. (10) and introducing vector $\mathbf{b} = (\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{v}_i$, one can express the absorbed power (38) as

$$\Phi = \tfrac{1}{2}\mathbf{b}^*\mathbf{M}\mathbf{b}, \quad \mathbf{M} = (\mathbf{Z}_r^* + \mathbf{Z}^*)^{-1}\mathbf{R}(\mathbf{Z}_r + \mathbf{Z})^{-1}. \tag{39}$$

The problem of the best absorber is now reformulated as follows: find such an impedance matrix $\mathbf{Z}$ that renders maximum to Φ or, in other words, for which the Hermitian form (39) is stationary. Giving matrix $\mathbf{Z}$ a variation $\Delta\mathbf{Z} = \Delta\mathbf{R} + \mathbf{i}\Delta\mathbf{X}$, $\Delta\mathbf{R}$ and $\Delta\mathbf{X}$ being arbitrary symmetric real-valued matrices of small matrix norm, one can write the stationary condition in the form:

$$\Delta\Phi = \Phi(\mathbf{Z} + \Delta\mathbf{Z}) - \Phi(\mathbf{Z}) = \tfrac{1}{2}\mathbf{b}^*\Delta\mathbf{M}\mathbf{b} = 0.$$

Considering the matrices $\Delta\mathbf{R}$ and $\Delta\mathbf{X}$ as independent variations of $\mathbf{Z}$ and going through the matrix algebra, one can find that $\Delta\mathbf{M}$ is null matrix if the absorber impedance matrix $\mathbf{Z}$ is equal to the Hermitian conjugate of the radiation impedance matrix:

$$\mathbf{Z} = \mathbf{Z}_r^* \quad \text{or} \quad \mathbf{R} = \mathbf{R}_r,\ \mathbf{X} = -\mathbf{X}_r. \tag{40}$$

By analyzing variations of the absorbed power in the vicinity of Eq (40), it can be shown that this stationary value is the maximum value. A body with the surface impedance matrix (40) is, thus, the best absorber and, among variety of bodies of the same configuration, it absorbs maximum of the incident field energy. This maximum value is equal to

$$\Phi_{\max} = \tfrac{1}{8}\mathbf{b}^*\mathbf{R}_r^{-1}\mathbf{b} = \tfrac{1}{8}\mathbf{p}_i^*(\mathbf{I} + \mathbf{Y}_i^*\mathbf{Z}_r^*)\mathbf{R}_r^{-1}(\mathbf{I} + \mathbf{Z}_r\mathbf{Y}_i)\mathbf{p}_i, \tag{41}$$

where $\mathbf{p}_i$ is a vector of the pressure amplitudes of the incident field.

Some general conclusions can be drawn directly from Eq. (40). First, the properties of the best absorber do not depend on the incident field and are fully determined by the acoustic environment, i.e. by the radiation impedances. Second, a body with a locally reacting surface cannot generally be the best absorber. It follows from the fact that the radiation impedance matrix is not diagonal in most real situations. The best absorber should, hence, have a surface of extended reaction (see Example 2). Condition (40) also means that the best absorber should be of the resonant type: its reactances must compensate the reactances of the surrounding medium, $\mathbf{X} + \mathbf{X}_r = 0$. As a result, the best absorber is at the same time a good scatterer: as proved in Ref. [17], the absorbed power is equal to the scattered power. Besides, the resistances of the best absorber are equal to the radiation resistances—see first Eq. (40). This is similar to the optimal load condition in electric circuit theory [18]. Eq. (40) can, thus, be interpreted as a matching condition of the absorber and acoustic environment. Note that the link between the amount of the absorbed power and the radiation impedance has also been reported earlier, though is simplified 1-D form, in the acoustic literature [5,19,20].

5.2. Example 1: a gas bubble in a liquid

Consider absorption of the sound energy by a gas bubble in a liquid which is, under certain conditions, a rare example of natural best absorbers. The bubble is assumed to be a pulsating sphere of a small radius a having thus one vibratory degree of freedom. The specific radiation impedance z_r and internal impedance z_i of the sphere in the free space are [5]

$$z_r = -i\rho c \frac{h_0(x)}{h_1(x)} = \rho c\left(\frac{x^2}{1+x^2} - i\frac{x}{1+x^2}\right),$$
$$z_i = i\rho c \frac{j_0(x)}{j_1(x)} = \rho c \frac{ix \sin x}{\sin x - x\cos x}, \tag{42}$$

where j_m and h_m are the spherical Bessel and Hankel functions of order m; k, $x = ka$ and ρc are the wavenumber and characteristic impedance of the liquid. The specific impedance z of the corresponding best absorbing pulsating sphere of radius a is, according to Eq. (40), the complex conjugate of the radiation impedance, $z = \bar{z}_r$. Since the radiation reactance, $\mathrm{Im}(z_r)$, is negative, i.e. mass-controlled, the reactance of the best spherical absorber should be spring-controlled, the spring stiffness being dependent on frequency as $x^2/(1+x^2)$.

In Fig. 1, the solid line corresponds to the relative absorption cross-section, i.e. the absorbed sound power of the best spherical absorber normalized with the incident power $\Phi_0 = \pi a^2 |p_i^2|/2\rho c$. Other curves in Fig. 1 correspond to absorption of a gas bubble (more exactly, an air bubble in water). Impedance z of the pulsating gas bubble has the form of z_i in Eq. (42) with liquid parameters ρc and x replaced by $(\rho c)_{\mathrm{gas}} = 2.8 \times 10^{-4} \rho c$ and $x_{\mathrm{gas}} = 4.36x$. In the frequency range of our concern, x is small and the reactance of the air bubble is spring-controlled giving, together with mass of the entrained water, the natural frequency $x_0 = 0.014$. Dashed line 1 in Fig. 1 corresponds to the particular bubble of radius $a = 3.3$ mm whose internal damping (due to heat transfer and viscosity) is equal to the damping due to radiation. In this case, both conditions (40) are satisfied and the air bubble in water behaves like the true best absorber. If the bubble radius is smaller or greater than 3.3 mm, the internal damping of the bubble becomes greater or smaller than the radiation damping at $x = x_0$, and the absorbed power decreases (dashed lines 2 and 3 in Fig. 1). Note that the relative cross-section of the corresponding black body of Kirchhoff is equal to unity and for the matched sphere that has the specific impedance ρc it is equal to four, i.e. three orders of magnitude smaller than that of the best absorber.

5.3. Example 2: two interacting gas bubbles in a liquid

To illustrate the effects of coupling between absorbers on the amount of the absorbed energy consider now two identical pulsating gas bubbles separated by distance $l = \gamma a$ in the unbounded liquid. The specific radiation impedance matrix of two identical spheres of radius a with account of multiple scattering effects is

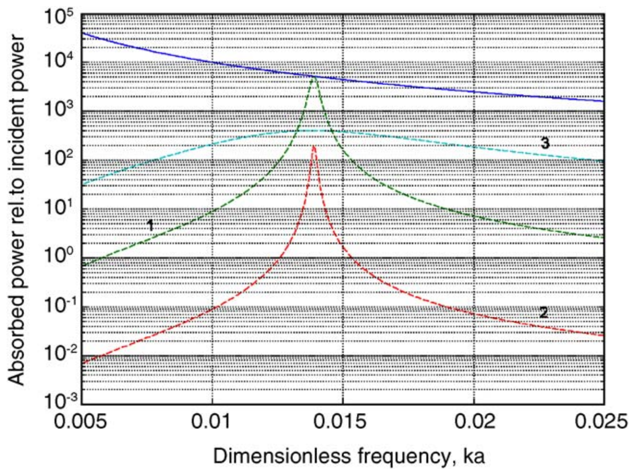

Fig. 1. Sound power absorbed by the best spherical absorber of radius a (solid line) and by a pulsating air bubble in water (dashed lines). Curve 1 corresponds to the optimal internal damping of the bubble that is equal to the resonant radiation damping; curves 2 and 3 correspond to internal damping, which is equal to 0.01 and 50 of the optimal value. The absorbed power is normalized with the incident power $\pi a^2 |p_i|^2/2\rho c$.

(the derivation see in Appendix A)

$$\mathbf{Z}_r = \frac{z_r}{1-d^2}\begin{bmatrix} 1+d(e-d) & e \\ e & 1+d(e-d) \end{bmatrix},$$
$$d = -h_0(\gamma x)j_1(x)/h_1(x) = xj_1(x)\exp(\mathrm{i}x(\gamma-1))/\gamma(1-\mathrm{i}x),$$
$$e = -ih_0(\gamma x)/x^2h_0(x)h_1(x) = \exp(\mathrm{i}x(\gamma-2))/\gamma(1-\mathrm{i}x), \tag{43}$$

where $x = ka$, z_r is the specific radiation impedance of an isolated sphere given in Eq. (42). This matrix relates the vector of two pressure amplitudes on the surface of the spheres to the vector of the radial velocity amplitudes of the surfaces. The off-diagonal elements of matrix (43) characterize interaction of the two spheres through the medium. The matrix $\mathbf{Z}_i$ of the internal impedances of the two spheres is diagonal both diagonal elements being equal to z_i in Eq. (42).

The absorbed power value being a Hermitian form with matrix $\mathbf{M}$ in Eq. (39) depends on the relation between the pressure amplitudes p_{i1} and p_{i2} of the incident field at the surfaces of the spheres. The Hermitian form and, hence, the absorbed power is restricted by the bounds

$$\lambda_2 \leqslant \frac{\mathbf{b}^*\mathbf{M}\mathbf{b}}{\mathbf{b}^*\mathbf{b}} \leqslant \lambda_1 \quad \text{or} \quad \frac{1}{2}\lambda_2\|\mathbf{b}\|^2 \leqslant \Phi \leqslant \frac{1}{2}\lambda_1\|\mathbf{b}\|^2,$$

where λ_1 and λ_2 are the eigenvalues of matrix $\mathbf{M}$, and vector $\mathbf{b}$ is equal to

$$\mathbf{b} = (\mathbf{Z}_r + \mathbf{Z}_i)\mathbf{v}_i = -(\mathbf{I} + \mathbf{Z}_r\mathbf{Y}_i)\mathbf{p}_i, \quad \mathbf{p}_i = [p_{i1}, p_{i2}]^{\mathrm{T}}.$$

Since $\mathbf{M}$ is a Hermitian matrix, the eigenvalues are real-valued (and non-negative).

For the best absorbing two-sphere system, the specific impedance matrix is $\mathbf{Z} = \mathbf{Z}_r^*$, so that the matrix $\mathbf{M}$ is equal to $\mathbf{M} = (\mathrm{Re}\,\mathbf{Z}_r)^{-1}/4$. Solid lines in Fig. 2 show the minimum and maximum of the power absorbed by this system, and the dashed-dotted line presents that of two isolated (non-interacting, e.g. located at a large distance) best spherical absorbers.

It is seen from Fig. 2 that the power absorbed by the best two-sphere absorber may be smaller or greater than that of two independent best spherical absorbers. This is a consequence of coupling between the spheres that is described by the off-diagonal elements of the radiation matrix (43). These elements act in such a way that the absorbed power is maximum when the incident field pressure amplitudes are of opposite sign ($p_{i2} = p_{i1}$), and the power is minimum when the amplitudes are identical ($p_{i2} = -p_{i1}$). For all other combinations of the pressure amplitudes of the incident field, the absorbed power is of an intermediate value.

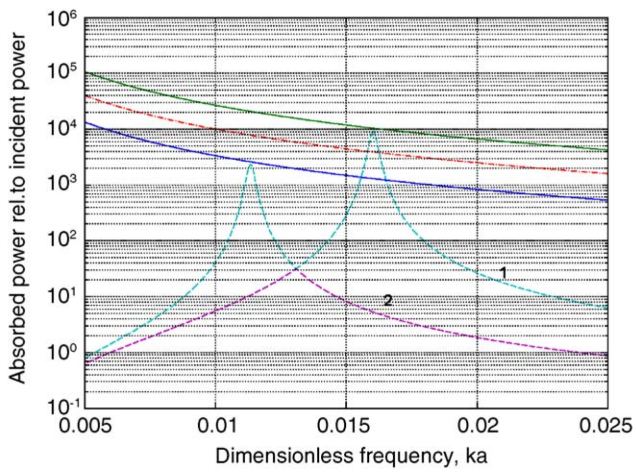

Fig. 2. The upper and lower bounds of the sound power absorbed by the best two-sphere absorber (solid lines) and by two non-interacting best spherical absorbers (dashed-dotted line). Dashed lines 1 and 2 correspond to the sound power absorbed by two interacting air bubbles in water that have a near optimal damping and located at distance $3a$ from each other. The power is normalized with the incident power $\pi a^2(|p_{i1}|^2 + |p_{i2}|^2)/2\rho c$.

When the distance l between the spheres increases, the coupling between the spheres diminishes, the difference between the minimum and maximum power values (i.e. between the solid lines in Fig. 2) vanishes. These curves coincide with the dashed-dotted line when the spheres become fully uncoupled.

One more effect of coupling is illustrated by the dashed lines in Fig. 2. These two curves correspond to the sound power that can be absorbed by two air bubbles in water. The individual eigenfrequencies of the bubbles are chosen to be identical and equal to $x_0 = 0.014$ (as in Fig. 1). Coupling between the bubbles splits the eigenfrequencies: one becomes higher than x_0 and the other lower than x_0. The power absorbed by these interacting bubbles depends on the pressure form of the incident field. When the incident field is anti-symmetric ($p_{i2} = -p_{i1}$), the dependence of the absorbed power on frequency is described by the dashed line 1. When it is symmetric ($p_{i2} = p_{i1}$), the curve 2 works. When any other combination of p_{i1} and p_{i2} takes place, the value of the absorbed power lies between these two dashed lines. Note that these curves correspond to the almost optimal internal damping coinciding with the radiation damping at the frequency x_0. If the internal damping is not optimal, the absorbed power decreases in a manner shown in Fig. 1.

When the number of bubbles is more than two, the role of coupling becomes even more important increasing the maximum achievable value of the absorbed sound power and widening the gap between the upper and lower bounds.

All these results are obviously valid not only for bubbles but for any other type of absorbing bodies or devices, e.g. for absorber areas on room walls. As follows from the above results, absorption efficiency of several commonly used absorbers can be noticeably increased by introducing a proper coupling between them.

6. Ideal scatterer

The power of sound scattered by a body in a fluid is, by definition, equal to

$$F = \tfrac{1}{2}\mathrm{Re}(\mathbf{v}_s^*\mathbf{p}_s) = \tfrac{1}{2}\mathbf{v}_i^*\mathbf{Q}^*\mathbf{R}_r\mathbf{Q}\mathbf{v}_i, \tag{44}$$

where $\mathbf{R}_r = \mathrm{Re}(\mathbf{Z}_r)$ is a matrix of the radiation resistances and $\mathbf{Q}$ is the scattering matrix given in Eqs. (9), (10). Analysis of this quantity, similar to that of the preceding section, has shown that it is stationary (minimum) only when the matrix of the body in vacuo impedances is equal to the fluid internal impedance matrix, $\mathbf{Z} = \mathbf{Z}_i$. In this case, the scattered power is zero and the body is acoustically transparent—see the next section. The largest value, though non-stationary, the scattered power (44) reaches when the resistance matrix of the body is the null matrix (the body is lossless) and its reactance matrix is equal to

$$\mathbf{R} = 0, \quad \mathbf{X} = -\mathbf{X}_r - \mathbf{R}_r(\mathbf{X}_i + \mathbf{X}_r)^{-1}\mathbf{R}_r. \tag{45}$$

The most efficient (ideal) passive scatterer is, thus, of the resonant type (just as the best absorber—see Eq. (40)), but with no damping. The largest possible value of the sound power that can be scattered by a passive scatterer follows from Eqs. (44) and (45) and equals

$$F_{\max} = \frac{1}{2}\mathbf{v}_i^*[\mathbf{R}_r + (\mathbf{X}_i + \mathbf{X}_r)\mathbf{R}_r^{-1}(\mathbf{X}_i + \mathbf{X}_r)\mathbf{v}_i. \tag{46}$$

This value is four times greater than the sound power (41) absorbed by the best absorber of the same geometry. Air bubble in water is an example of the almost ideal scatterer. At resonance, it scatters the sound power that is close to the largest value (46) if the bubble radius is greater than 3.3 mm and its natural frequency falls below 1 kHz so that the energy loss associated with viscosity and heat transfer is small compared to the sound radiation energy loss.

7. Acoustically transparent body

It is apparent from Eq. (10) that a body of volume V does not scatter sound and its scattering matrices are zero matrices, $\mathbf{Q} = \mathbf{S} = 0$, if its matrix of the in vacuo surface impedances coincides with the matrix of the fluid internal impedances

$$\mathbf{Z} = \mathbf{Z}_i. \tag{47}$$

This result physically means that it is the body surface reaction that matters in scattering sound, the internal structure of the body being insignificant. It implies that arbitrary body can be made to be non-scattering, i.e. acoustically transparent, with the help of a coating whose surface reaction to external sound fields is matched to that of the fluid displaced by the body.

It follows from Eq. (47) that it is impossible to construct a non-scattering passive coating that is made of a locally reacting material. It is because the fluid-filled volume V is a vibratory system with low damping and high quality resonances. A local excitation at the surface of such a system would produce a global reaction of all parts of the system. In other words, the system has an essentially extended reaction, so that its matrix of the surface impedances, $\mathbf{Z}_i$, is far from being diagonal (note that the impedance matrix of a locally reacting system is always diagonal).

One way of implementing equality (47) in practically interesting cases is to use active structures, e.g. smart skins. Such an active structure should be of the global type in that the active force applied to a certain part of the body surface should be controlled by the field quantities measured at other parts of the surface. A global feedback control system of this type is proposed in paper [21]. It represents a coating that integrates N structural actuators, N structural sensors (accelerometers), and N surface pressure sensors–by one at each interface element ΔA_n. It is shown in Ref. [21] that the vector of active forces $\mathbf{f}_a = [f_{a1}, \ldots, f_{aN}]^{\mathrm{T}}$ applied to the interface A may be controlled either by the total field velocities measured by the structural sensors

$$\mathbf{f}_a = (\mathbf{Z} - \mathbf{Z}_i)\mathbf{v}, \tag{48}$$

or by the total pressure field amplitudes measured on the interface by acoustic sensors

$$\mathbf{f}_a = (\mathbf{I} - \mathbf{Z}\mathbf{Y}_i)\mathbf{p}. \tag{49}$$

Both control laws, Eqs. (48) and (49), provide the full compensation of the scattered field component, $\mathbf{p}_s$ and $\mathbf{v}_s$, thus, making the body acoustically transparent.

When the body is also a source of its own radiation field, it is possible to compensate both the scattered field component and the radiation component using the same actuators. The control law, in this case, is more complicated

$$\mathbf{f}_a = -\boldsymbol{\sigma} - \mathbf{Z}_i\mathbf{v}, \tag{50}$$

where $\boldsymbol{\sigma}$ is a vector of the normal stresses in the body which should also be measured on the body surface A. The active force (50) is controlled by a pair of quantities, the normal stresses and the total normal velocities, measured all over the body surface, instead of one measured vector quantity needed to suppress a single field component—the radiation component or the scattered component (as in Eqs. (48) and (49)).

The problem of an acoustically transparent body has been formulated and solved by several authors in 1960s of the last century. All the solutions available so far, in particular those obtained by the methods of Maluzhinetz and JMC (abbreviation of Jessel—Mangiant–-Canevet), are based on the Huygens principle and the use of acoustic sensors of both types (pressure and velocity) and acoustic actuators both of the monopole and dipole types placed on several closed surfaces enveloping the body. The reviews of the literature on the subject can be found in Ref. [21,22]. The solution, based on the impedance approach and the use of structural sensors and actuators, proposed in Ref. [21], is an alternative to the existing solutions, which, in the present author's opinion, is more appropriate for practical realization.

As an example, consider active suppression of sound scattered by an empty thin circular cylindrical shell of infinite extent in an unbounded liquid. The scattering problem as well as the problem of forced coupled vibrations of the shell–liquid system is tractable analytically and well studied in the literature. Therefore, all the mathematical details are dropped below, and an interested reader can address one of the available textbooks on the subject, say, Ref. [5]. It is only worth mentioning that, in this paper, the author employed the Donnell–Mushtari equations and the commonly used technique of expanding the vibroacoustic fields in circumferential modes and cylindrical waves, restricting the consideration to scattering of plane waves normally incident on the shell.

Fig. 3 depicts the relative scattering cross-section of the shell, i.e. power F of the scattered field component normalized with the incident power $F_{\mathrm{inc}} = a|p_i|^2/\rho c$, as a function of frequency. Here p_i is the amplitude of the incident field, a is the radius of the shell, ρc and k are the characteristic impedance and wavenumber of the

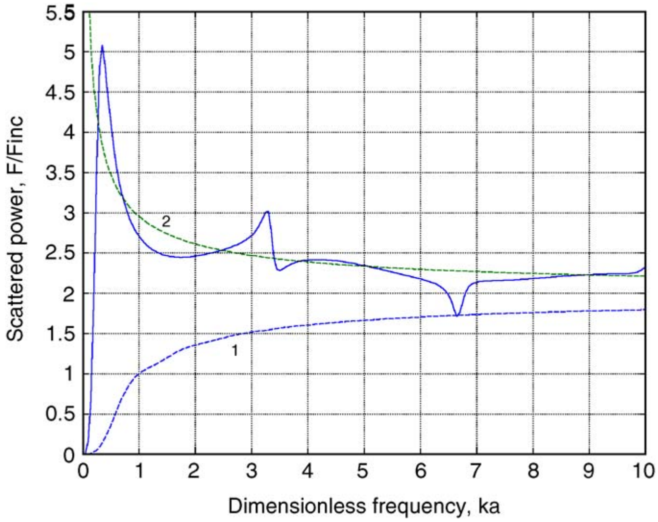

Fig. 3. Power per unit length, F, of sound scattered by an infinite circular cylindrical shell of steel in water as described in Section 7 (solid line), by a rigid cylinder (dashed line 1), and by a soft cylinder (dashed line 2). It is normalized with the power of an incident plane wave, $F_{\text{inc}} = a|p_i|^2/\rho c$, that falls normally on the cylinder.

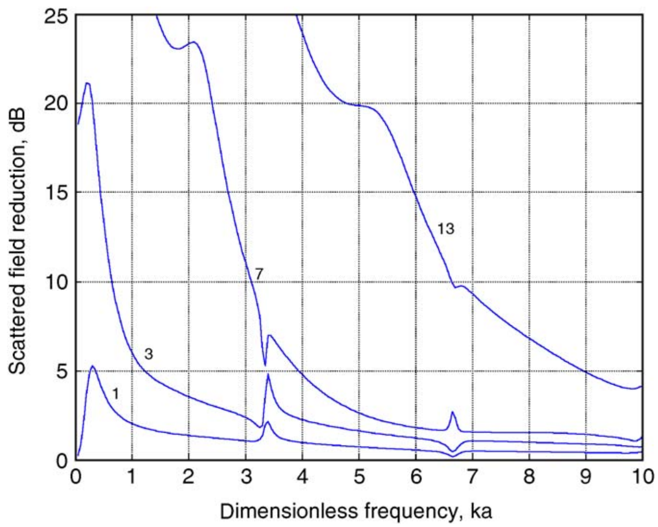

Fig. 4. Reduction in dB of the sound field scattered by a thin cylindrical shell due to N active line forces applied to the shell in radial direction. The number N is indicated near each curve.

liquid, the thickness of the shell is $h/a = 0.002$; the loss factor of the shell is 0.01. The sound power scattered by the shell (solid line in Fig. 3) has a distinct maximum in the low-frequency range associated with resonance of the breathing mode. For comparison, the relative scattering cross-sections of the rigid cylinder of radius a and of the cylinder with the pressure-released surface is shown by dashed lines.

Fig. 4 shows the efficiency of active control of the scattered field by means of several active line forces (48) applied to the shell equidistantly on the circumference. Each line force is uniformly distributed on a straight line along the shell and act on it in the radial direction. Figures written near the curves in Fig. 4 indicate the number of the active forces used. It is seen from Fig. 4 that the efficiency of the control is strongly dependent on the number of the active forces N: the larger N, the wider the frequency band of reduction of the scattered field and the higher the efficiency. The frequency band of reduction is limited from above by the frequency $ka = N/2$ at which the distance between the active forces is half the acoustic wavelength.

8. Summary

A new theoretical solution to the sound scattering problem is proposed. It is applicable to elastic scatterers of any configuration in a fluid medium that may be bounded and inhomogeneous. The solution is written in terms of three impedance matrices that are defined with respect to the scatterer–medium interface and fully characterize the scattered field. General properties of the solution, such as symmetry of the scattering matrices, are studied. Several new useful representations of the scattered field are also presented.

The proposed solution is applied to three known practical problems of sound scattering and absorption for which some new results are obtained. These are the problems of the best absorber, of the ideal scatterer, and of an acoustically transparent body. New solutions to these problems are formulated in terms of the in vacuo surface impedances of the scatterer.

In practice, these solutions can be implemented by means of thin active coatings placed on the scatterer surface and using the active impedance matching method. Some results on matching of the local impedances are available in the literature—see, e.g. Refs. [23–26]. The matching needed here should be of the global type, that is to say, matching of the impedance matrices–see Ref. [27]. This is a rather difficult for practical realization procedure, which nevertheless seems to have no realistic alternative.

Acknowledgment

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Appendix A. Radiation impedance matrix of two small pulsating spheres in an unbounded medium

The radiation impedance matrix (43) of two small spheres (gas bubbles) in a liquid is derived in this appendix with taking into account multiple scattering. The spheres are assumed to be identical, their radius a is small in terms of the acoustic wavelength. Both spheres vibrate only in pulsating mode, all other types of their movement are excluded from the consideration. The ambient liquid is assumed to be unbounded. The distance between the sphere centers is $l = \gamma a$, $\gamma > 2$.

The following commonly used method of obtaining impedance matrices is employed here. In case of two pulsating spheres, it consists in solving two auxiliary boundary value problems. In the first problem, sphere 1 is assumed to pulsate with a prescribed surface velocity v_1 while the surface of the sphere 2 is blocked, $v_2 = 0$. For this configuration, the pressure response amplitudes, p_1 and p_2, on the sphere surfaces are found. The ratios p_1/v_1 and p_2/v_1 represent, by definition, the first column of the sought radiation impedance matrix (43). The second column of the matrix corresponds to the solution of the second auxiliary boundary value problem, in which the surface of the sphere 1 is blocked, $v_1 = 0$, and the sphere 2 pulsates with a prescribed radial velocity, v_2. In our case of identical spheres, the second column consists of the same two elements as the first column. For that reason, only a solution of the first auxiliary problem is presented in what follows. The resulting pressure amplitudes, p_1 and p_2, are computed by summing all the successively scattered multiple components.

Suppose first that the sphere 2 is removed from the fluid and the sphere 1 pulsates with a surface velocity v_1. Then the pressure field radiated by the sphere 1 is [5]

$$p(r) = v_1 z_r h_0(kr)/h_0(ka), \tag{A.1}$$

where z_r is the specific radiation impedance of a pulsating sphere in an unbounded fluid given in Eq. (42), r is the distance from the sphere center. On the surface of the sphere 1, the pressure amplitude is equal to

$$p_{10} = p(a) = v_1 z_r. \tag{A.2}$$

On the surface of a sphere of radius a located at distant $r = l = \gamma a$ from the first sphere, this field is equal to

$$\begin{aligned} p_{20i} &= p(l) j_0(ka) = p_{10}\delta, \\ \delta &= h_0(\gamma x) j_0(x)/h_0(x), \end{aligned} \tag{A.3}$$

where j_0 and h_0 are the spherical Bessel and Hankel functions, $x = ka$. Eq. (A.3) follows from the Helmholtz–Huygens integral representation [3] of the field (A.1)

$$p(r) = (v_1 z_r / h_0(x)) \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) j_n(ka) h_n(kr)$$

after neglecting all the types of movement but the pulsating one.

Field (A.3) serves as an incident pressure field for the second sphere. Let us now return the blocked sphere 2 at its place and compute the amplitudes of the scattered field and total field on the surface of the sphere 2 that correspond to the incident field (A.3). According to Eq. (9), they are equal to

$$\begin{aligned} p_{20s} &= S_{\mathrm{rig}} p_{20i} = p_{10} \delta S_{\mathrm{rig}}, \\ p_{20} &= p_{20i} + p_{20s} = p_{10} \delta (1 + S_{\mathrm{rig}}), \end{aligned} \tag{A.4}$$

where $S_{\mathrm{rig}} = z_r z_i^{-1}$ is the pressure scattering coefficient for a rigid sphere—see Eq. (18). Field (A.4) is the result of "the first scattering". The scattered component (A.4) serves, in its turn, as an incident field for the sphere 1. Its amplitude on the surface of the sphere 1 can be obtained similarly to Eq. (A.3) as

$$p_{11i} = p_{20s} \delta = p_{10} \delta^2 S_{\mathrm{rig}}. \tag{A.5}$$

The result of "the second scattering", i.e. the field scattered by the sphere 1, yields the following components of the scattered and total fields on the sphere 1

$$\begin{aligned} p_{11s} &= S_{\mathrm{rig}} p_{11i} = p_{10} d^2, \\ p_{11} &= p_{11i} + p_{11s} = p_1 ed, \end{aligned} \tag{A.6}$$

where d and e denote the expressions:

$$\begin{aligned} d &= \delta S_{\mathrm{rig}} = -\frac{h_0(\gamma x) j_1(x)}{h_1(x)}, \\ e &= \delta(1 + S_{\mathrm{rig}}) = -\frac{i h_0(\gamma x)}{x^2 h_0(x) h_1(x)}. \end{aligned} \tag{A.7}$$

Considering further the scattered component (A.6) as the incident field for the sphere 2, one can obtain the fields of "the third scattering" on the surface of the sphere 2

$$p_{21s} = p_{10} d^3, \quad p_{21} = p_{10} e d^2. \tag{A.8}$$

In the same manner field components of "the fourth scattering" can be obtained on sphere 1 as

$$p_{12s} = p_{10} d^4, \quad p_{12} = p_{10} e d^3. \tag{A.9}$$

Continuing this process and summing all the components of the multiple scattering (A.1)–(A.9), one can obtain the following resulting pressure amplitudes on the surface of the two spheres:

$$\begin{aligned} p_1 &= p_{10} + p_{11} + p_{12} + \cdots = p_{10} \frac{1 + d(e - d)}{1 - d^2}, \\ p_2 &= p_{20} + p_{21} + \cdots = p_{10} \frac{e}{1 - d^2}. \end{aligned} \tag{A.10}$$

Here p_{10} is given in Eq. (A.2), d and e are deciphered by Eq. (A.7). The first column of the radiation impedance matrix (43) is obtained from the amplitudes (A.10) as $[p_1/v_1; p_2/v_1]^{\mathrm{T}}$. The second column of matrix (43) is equal to $[p_2/v_1; p_1/v_1]^{\mathrm{T}}$.

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